3.10.1. Duality Problems

A. For each of the following formal sentences, state its **connective dual**.

1. $(P \oplus \top)$
2. $(P \oplus (P \leftrightarrow P))$
3. $(\top \rightarrow \bot)$
4. $((P \rightarrow P) \rightarrow P)$
5. $(Q \wedge (P \oplus Q))$
6. $(Q \oplus (P \oplus Q))$
7. $(P \vee (Q \,\%\, P))$

B. Build a truth table for each sentence in (A). Which of these sentences is a **semantic self-dual**?

C. We translated an "otherwise" sentence such as "If P, Q; otherwise R" as the conjunction of two conditionals.

If P, Q; otherwise R

$((P \rightarrow Q) \wedge (\sim P \rightarrow R))$

But suppose we introduce a single three-place connective "#" to express such a sentence.

If P, Q; otherwise R

$(P \,\#\, Q\, R)$

1. Show that the language $\{\#, \top, \bot\}$ **is expressively adequate**.[1]

(Hint: use the adequacy of the $\{\rightarrow, \sim\}$ language.)

2. Build a $\{\#, \top, \bot\}$ sentence logically equivalent to "$(P \leftrightarrow Q)$".

3. Build a $\{\#, \top, \bot\}$ sentence logically equivalent to "$(P \wedge Q)$".

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[1] Following the discussion in (Church 1956: 129-132) – though Church use the slightly different sentence form "P if Q; otherwise R," which translates as "$((Q \rightarrow P) \wedge (\sim Q \rightarrow R))$".

D. Suppose we introduce a new connective, "**$**," equivalent to the following.

Either not P and Q, or P and R

(P **$** Q R)

1. Translate and build a truth table for "Either not P and Q, or P and R," in order to state the **semantic rule** for the connective "**$**".

2. Show that the language {**$**, ⊤, ⊥} **is expressively adequate**.

(Hint: use the adequacy of the {%, ~} language)

3. Use **duality** to show that the {**$**, ⊤, ⊥} **language is expressively adequate**.

4. Build a {**$**, ⊤, ⊥} sentence logically equivalent to "(P ∧ Q)".

5. Build a {**$**, ⊤, ⊥} sentence logically equivalent to "(P ⊕ Q)".